Suggested Solution to Test 1[1]

1. (a) First, consider

$$\begin{aligned} \langle \mathbf{u}, (2,2,1)\rangle &= 0 \\ a(2) + (-1)(2) + 0(1) &= 0 \\ a &= 1 \end{aligned}$$

and

$$\begin{aligned} \|\mathbf{v}\| &= 3 \\ \sqrt{b^2 + (-1)^2 + b^2} &= 3 \\ 2b^2 + 1 &= 9 \\ b^2 &= 4 \\ b = 2 \quad \text{or} \quad b &= -2 \text{ (rejected as } b > 0) \end{aligned}$$

(b) From (a), we have $\mathbf{u} = (1, -1, 0)$ and $\mathbf{v} = (2, -1, 2)$.
Then, we have

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 2 & -1 & 2 \end{vmatrix} \\ &= -2\mathbf{i} - 2\mathbf{j} + \mathbf{k} \end{aligned}$$

Thus, we have

$$\langle \mathbf{w}, \mathbf{u} \times \mathbf{v}\rangle = 2(-2) + 1(-2) + (-1)(1) = -7.$$

(c) The area required is $\|\mathbf{u} \times \mathbf{v}\| = \sqrt{(-2)^2 + (-2)^2 + 1^2} = 3$.

(d) The volume of the tetrahedron generated by $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ is

$$\begin{aligned} &\frac{1}{3}\left(\frac{1}{2}\|\mathbf{u} \times \mathbf{v}\|\right)\left(\frac{|\langle \mathbf{w}, \mathbf{u} \times \mathbf{v}\rangle|}{\|\mathbf{u} \times \mathbf{v}\|}\right) \\ =& \frac{1}{6}\,|-7| \\ =& \frac{7}{6} \text{ cubic units} \end{aligned}$$

(e) The distance between $\mathbf{w}$ and the plane spanned by $\mathbf{u}$ and $\mathbf{v}$ is

$$\begin{aligned} d &= \frac{\text{Volume of the Tetrahedron}}{\frac{1}{3} \times (\text{Base area of the triangle spanned by } \mathbf{u} \text{ and } \mathbf{v})} \\ &= \frac{\frac{7}{6}}{\frac{1}{3} \cdot \frac{3}{2}} \\ &= \frac{7}{3} \end{aligned}$$

2. (a) The directional of the line segment is $(2, 7) - (-5, 2) = (7, 4)$.
Thus, we can parametrize the line segment by

$$\mathbf{r}(t) = (-5, 2) + t(7, 4) = (-5 + 7t, 2 + 4t),\ t \in (0, 1)$$

[1] If you have any problems or typos, please contact me via **maxshung.math@gmail.com**

(b) The Cartesian equation of the circle is

$$(x-7)^2+(y+5)^2=3^2$$

Letting

$$\begin{cases} x-7=3\cos t \\ y+5=3\sin t \end{cases}$$

for $t \in [0, 2\pi)$.
Thus, we can parametrize the circle by

$$\mathbf{r}(t)=(7+3\cos t, -5+3\sin t),\ t \in [0, 2\pi)$$

(c) Rewrite the equation of the ellipse as

$$\left(\frac{x-5}{3}\right)^2+\left(\frac{y-3}{2\sqrt{2}}\right)^2=\left(\sqrt{2}\right)^2$$

Letting

$$\begin{cases} \dfrac{x-5}{3}=\sqrt{2}\cos t \\ \dfrac{y-3}{2\sqrt{2}}=\sqrt{2}\sin t \end{cases}$$

for $t \in [0, 2\pi)$.
Thus, we can parametrize the ellipse by

$$\mathbf{r}(t)=\left(5+3\sqrt{2}\cos t, 3+4\sin t\right),\ t \in [0, 2\pi)$$

(d) Rewrite the equation of the hyperbola as

$$\frac{9x^2}{4}-\frac{y^2}{18}=1$$
$$\left(\frac{x}{2/3}\right)^2-\left(\frac{y}{3\sqrt{2}}\right)^2=1$$

Letting

$$\begin{cases} \dfrac{x}{2/3}=-\cosh t \\ \dfrac{y}{3\sqrt{2}}=\sinh t \end{cases}$$

for $t \in \mathbb{R}$.
Thus, we can parametrize the hyperbola by

$$\mathbf{r}(t)=\left(-\frac{2}{3}\cosh t, 3\sqrt{2}\sinh t\right),\ t \in \mathbb{R}$$

Note. We may also use $\sec$ and $\tan$ to parametrize it by letting

$$\begin{cases} \dfrac{x}{2/3}=\sec t \\ \dfrac{y}{3\sqrt{2}}=\tan t \end{cases} \quad \text{for} \quad t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

An alternative parametrization for the hyperbola is

$$\mathbf{r}(t)=\left(\frac{2}{3}\sec t, 3\sqrt{2}\tan t\right),\ t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

3. (a) Since

$$\begin{aligned}\mathbf{r}'(t) &= \left(\sec^2 t - 1, \tan t \cdot \sec^2 t - \frac{\sec t \tan t}{\sec t}\right)\\ &= \left(\tan^2 t, \tan t(\sec^2 t - 1)\right)\\ &= \left(\tan^2 t, \tan^3 t\right)\end{aligned}$$

and $\tan t \neq 0$ for all $0 < t < \frac{\pi}{4}$ and so $\mathbf{r}'(t) \neq \mathbf{0}$.
Thus, $\mathbf{r}(t)$ is a regular parametrized curve.

(b) From (a), note that

$$\begin{aligned}\|\mathbf{r}'(t)\| &= \sqrt{(\tan^2 t)^2 + (\tan^3 t)^2}\\ &= \tan^2 t\sqrt{1 + \tan^2 t}\\ &= (\sec^2 t - 1)\sec t\\ &= \sec^3 t - \sec t\end{aligned}$$

Thus, the arclength of $\mathbf{r}(t)$ over $\left(0, \frac{\pi}{4}\right)$ is

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \|\mathbf{r}'(t)\| dt &= \int_0^{\frac{\pi}{4}} (\sec^3 t - \sec t)\, dt\\ &= \left.\frac{\sec t \tan t}{2}\right|_0^{\frac{\pi}{4}} + \frac{1}{2}\int_0^{\frac{\pi}{4}} \sec t\, dt - \int_0^{\frac{\pi}{4}} \sec t\, dt\\ &= \frac{\sqrt{2}}{2} - 0 - \frac{1}{2}\int_0^{\frac{\pi}{4}} \sec t\, dt\\ &= \frac{\sqrt{2}}{2} - \frac{1}{2}\ln|\sec t + \tan t|\Big|_0^{\frac{\pi}{4}}\\ &= \frac{\sqrt{2}}{2} - \frac{1}{2}\ln\left(\sqrt{2} + 1\right)\end{aligned}$$

4. (a) By direct computation, we have

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2\\ &= \left(\frac{e^x + e^{-x} + e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x} - e^x + e^{-x}}{2}\right)\\ &= e^x \cdot e^{-x}\\ &= 1\end{aligned}$$

for any $x \in \mathbb{R}$.

(b) Since

$$\begin{aligned}x &= \frac{e^y - e^{-y}}{2}\\ 2x &= e^y - e^{-y}\\ e^{2y} - 2xe^y - 1 &= 0\end{aligned}$$

$$e^y = \frac{2x + \sqrt{(2x)^2 - 4(1)(-1)}}{2} \quad \text{or} \quad e^y = \frac{2x - \sqrt{(2x)^2 - 4(1)(-1)}}{2} \text{ (rejected } \because < 0)$$

$$\begin{aligned}e^y &= \frac{2x + \sqrt{4(x^2 + 1)}}{2}\\ y &= \ln\left(x + \sqrt{x^2 + 1}\right)\end{aligned}$$

for any $x \in \mathbb{R}$.

(c) (i) $\mathbf{r}'(t) = (1, \sinh t)$ and $\|\mathbf{r}'(t)\| = \sqrt{1 + \sinh^2 t} = \cosh t$.

(ii) The arc-length function of $\mathbf{r}(t)$ is

$$\begin{aligned} s &= \int_0^t \|\mathbf{r}'(u)\| \, du \\ &= \int_0^t \cosh u \, du \\ &= \sinh t \end{aligned}$$

Therefore, we have $t = \sinh^{-1} s = \ln\left(s + \sqrt{1 + s^2}\right)$.
Thus, the arclength parametrization of **catenary** is

$$\begin{aligned} \mathbf{r}(s) &= \left(\ln\left(s + \sqrt{1+s^2}\right), \cosh(\sinh^{-1} s)\right) \\ &= \left(\ln(s + \sqrt{1+s^2}), \sqrt{1 + \sinh^2(\sinh^{-1} s)}\right) \\ &= \left(\ln(s + \sqrt{1+s^2}), \sqrt{1+s^2}\right), \quad s > 0 \end{aligned}$$

5. (a) Note that

$$\begin{aligned} & \left\| \left(\frac{A + A^T}{2}\right) \mathbf{u} \right\|^2 + \left\| \left(\frac{A - A^T}{2}\right) \mathbf{u} \right\|^2 \\ &= \mathbf{u}^T \left(\frac{A + A^T}{2}\right)^T \left(\frac{A + A^T}{2}\right) \mathbf{u} + \mathbf{u}^T \left(\frac{A - A^T}{2}\right)^T \left(\frac{A - A^T}{2}\right) \mathbf{u} \\ &= \mathbf{u}^T \left(\frac{A + A^T}{2}\right)^2 \mathbf{u} - \mathbf{u}^T \left(\frac{A - A^T}{2}\right)^2 \mathbf{u} \\ &= \mathbf{u}^T \left[\frac{A^2 + AA^T + A^TA + (A^T)^2}{4} - \frac{A^2 - AA^T - A^TA + (A^T)^2}{4}\right] \mathbf{u} \\ &= \mathbf{u}^T \left(\frac{AA^T + A^TA}{2}\right) \mathbf{u} \\ &= \frac{1}{2}\mathbf{u}^T AA^T \mathbf{u} + \frac{1}{2}\mathbf{u}^T A^T A \mathbf{u} \\ &= \frac{1}{2}(A^T\mathbf{u})^T (A^T\mathbf{u}) + \frac{1}{2}(A\mathbf{u})^T (A\mathbf{u}) \\ &= \frac{1}{2}\|A^T\mathbf{u}\|^2 + \frac{1}{2}\|A\mathbf{u}\|^2 \end{aligned}$$

(b) Since $\dfrac{A + A^T}{2}$ is symmetric, and left multiplying $\mathbf{u}^T$ on both sides, then

$$\begin{aligned} \mathbf{u}^T \left(\frac{A + A^T}{2}\right)^2 \mathbf{u} &= \mathbf{u}^T \mathbf{0} \\ \mathbf{u}^T \left(\frac{A + A^T}{2}\right)^T \left(\frac{A + A^T}{2}\right) \mathbf{u} &= 0 \\ \left\langle \left(\frac{A + A^T}{2}\right) \mathbf{u}, \left(\frac{A + A^T}{2}\right) \mathbf{u} \right\rangle &= 0 \\ \left\| \left(\frac{A + A^T}{2}\right) \mathbf{u} \right\|^2 &= 0 \end{aligned}$$

Thus, we have $\left(\dfrac{A + A^T}{2}\right) \mathbf{u} = \mathbf{0}$, and follows that $A^T\mathbf{u} = -A\mathbf{u}$.

Alternative Solution.
From (a), if $\left(\dfrac{A+A^T}{2}\right)^2\mathbf{u}=\mathbf{0}$, then

$$\begin{aligned}
\left\|\left(\frac{A-A^T}{2}\right)\mathbf{u}\right\|^2 &= \frac{1}{2}\|A\mathbf{u}\|^2+\frac{1}{2}\|A^T\mathbf{u}\|^2\\
\mathbf{u}^T\left(\frac{A-A^T}{2}\right)^T\left(\frac{A-A^T}{2}\right)\mathbf{u} &= \frac{1}{2}\mathbf{u}^TA^TA\mathbf{u}+\frac{1}{2}\mathbf{u}^TAA^T\mathbf{u}\\
\mathbf{u}^T\left[-\frac{A^2-AA^T-A^TA+(A^T)^2}{4}-\frac{1}{2}A^TA-\frac{1}{2}AA^T\right]\mathbf{u} &= 0\\
-\mathbf{u}^T\left(\frac{A^2+AA^T+A^TA+(A^T)^2}{4}\right)\mathbf{u} &= 0\\
\mathbf{u}^T\left(\frac{A+A^T}{2}\right)^T\left(\frac{A+A^T}{2}\right)\mathbf{u} &= 0\\
\left\langle\left(\frac{A+A^T}{2}\right)\mathbf{u},\left(\frac{A+A^T}{2}\right)\mathbf{u}\right\rangle &= 0\\
\left\|\left(\frac{A+A^T}{2}\right)\mathbf{u}\right\|^2 &= 0
\end{aligned}$$

Thus, we have $\left(\dfrac{A+A^T}{2}\right)\mathbf{u}=\mathbf{0}$, and follows that $A^T\mathbf{u}=-A\mathbf{u}$.

(c) Proceed similar as part (b), we have

$$\begin{aligned}
\mathbf{v}^T\left(\frac{A-A^T}{2}\right)^2\mathbf{v} &= \mathbf{v}^T\mathbf{0}\\
-\mathbf{v}^T\left(\frac{A-A^T}{2}\right)^T\left(\frac{A-A^T}{2}\right)\mathbf{v} &= 0\\
\left\|\left(\frac{A-A^T}{2}\right)\mathbf{v}\right\|^2 &= 0
\end{aligned}$$

Therefore, it follows that $\dfrac{A-A^T}{2}\mathbf{v}=\mathbf{0}$ for any $\mathbf{v}\in\mathbb{R}^3\setminus\{\mathbf{0}\}$, that is

$$A^T\mathbf{v}=A\mathbf{v}$$

By taking $\mathbf{v}$ as $\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\1\end{pmatrix}$ respectively, it follows that

$$\begin{aligned}
\left(\ A^T\begin{pmatrix}1\\0\\0\end{pmatrix}\ \middle|\ A^T\begin{pmatrix}0\\1\\0\end{pmatrix}\ \middle|\ A^T\begin{pmatrix}0\\0\\1\end{pmatrix}\ \right) &= \left(\ A\begin{pmatrix}1\\0\\0\end{pmatrix}\ \middle|\ A\begin{pmatrix}0\\1\\0\end{pmatrix}\ \middle|\ A\begin{pmatrix}0\\0\\1\end{pmatrix}\ \right)\\
A^T\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} &= A\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\\
A^T &= A
\end{aligned}$$

By definition, A is symmetric and thus the claim is correct.
Remark. Taking any three linearly independent column vectors are acceptable, need not to be orthonormal basis for $\mathbb{R}^3$.

6. (a) Since $\|\mathbf{r}(t)-\mathbf{c}\| = C$, where C is a non-zero constant for any $t \in [a,b]$ and $\mathbf{r}(t) \in \mathbb{R}^3$ which is a space curve.
Thus $\mathbf{r}(t)$ is a closed curve lying on the sphere centered at $\mathbf{c}$ with radius C.

(b) (i) From (a), differentiating $\|\mathbf{r}(t)-\mathbf{c}\|^2 = C^2$ with respect to t on both sides,

$$\begin{aligned} \frac{d}{dt}\langle \mathbf{r}(t)-\mathbf{c}, \mathbf{r}(t)-\mathbf{c}\rangle &= \frac{d}{dt}C^2 \\ \left\langle \mathbf{r}(t)-\mathbf{c}, \mathbf{r}'(t)-\frac{d}{dt}\mathbf{c}\right\rangle + \left\langle \mathbf{r}'(t)-\frac{d}{dt}\mathbf{c}, \mathbf{r}(t)-\mathbf{c}\right\rangle &= 0 \\ \langle \mathbf{r}(t)-\mathbf{c}, \mathbf{r}'(t)\rangle &= 0 \end{aligned}$$

for any $t \in (a,b)$. (*We ignore the differentiability at the end points of $[a,b]$.)

(ii) Assume that $\mathbf{s}'(t)$ exists for any $t \in (a,b)$ and $\mathbf{s}'(t) \neq \mathbf{0}$, then

$$\begin{aligned} \langle \mathbf{s}(t)-\mathbf{c}, \mathbf{s}'(t)\rangle &= 0 \\ \frac{d}{dt}\langle \mathbf{s}(t)-\mathbf{c}, \mathbf{s}(t)-\mathbf{c}\rangle &= 0 \\ \langle \mathbf{s}(t)-\mathbf{c}, \mathbf{s}(t)-\mathbf{c}\rangle &= C' \end{aligned}$$

where C' is a nonnegative constant.
Therefore $\|\mathbf{s}(t)-\mathbf{c}\|$ is a constant independent of t, i.e. $\mathbf{s}(t)$ is a curve maintaining fixed distance from $\mathbf{c}$. Thus, the claim is agreed.

(c) From part (b)(i), we have

$$\langle \mathbf{r}(t)-\mathbf{c}, \mathbf{r}'(t)\rangle = 0$$

Also, as $\|\mathbf{r}'(t)\|$ is a constant, differentiating $\|\mathbf{r}'(t)\|^2$ with respect to t yields

$$\langle \mathbf{r}'(t), \mathbf{r}''(t)\rangle = 0$$

In $\mathbb{R}^3$, as $\mathbf{r}(t)-\mathbf{c}$ and $\mathbf{r}''(t)$ are both orthogonal to $\mathbf{r}'(t)$, so we have

$$(\mathbf{r}(t)-\mathbf{c}) \times \mathbf{r}''(t) = K(t)\mathbf{r}'(t)$$

for some scalar functions $K(t)$. Now, it remains to show $K(t)$ is a non-zero constant.
First, differentiating $\langle \mathbf{r}(t)-\mathbf{c}, \mathbf{r}'(t)\rangle = 0$ with respect to t, then

$$\begin{aligned} \langle \mathbf{r}(t)-\mathbf{c}, \mathbf{r}''(t)\rangle + \langle \mathbf{r}'(t), \mathbf{r}'(t)\rangle &= 0 \\ \langle \mathbf{r}(t)-\mathbf{c}, \mathbf{r}''(t)\rangle &= -\|\mathbf{r}'(t)\|^2 \neq 0 \end{aligned}$$

which is a negative constant. Therefore,

$$\begin{aligned} (K(t))^2 &= \frac{\|\mathbf{r}(t)-\mathbf{c}\|^2\|\mathbf{r}''(t)\|^2 - |\langle \mathbf{r}(t)-\mathbf{c}, \mathbf{r}''(t)\rangle|^2}{\|\mathbf{r}'(t)\|^2} \\ &> \frac{\|\mathbf{r}'(t)\|^4 - (-\|\mathbf{r}'(t)\|^2)^2}{\|\mathbf{r}'(t)\|^2} \\ &= 0 \end{aligned}$$

So, $K(t)$ is non-zero.
Moreover, we see that $K(t)$ is depending on $\|\mathbf{r}(t)-\mathbf{c}\|$, $\|\mathbf{r}''(t)\|$ and $\langle \mathbf{r}(t)-\mathbf{c}, \mathbf{r}''(t)\rangle$ are all constants, so $K(t)$ is constant independent of t.
Thus, we have

$$(\mathbf{r}(t)-\mathbf{c}) \times \mathbf{r}''(t) = k\mathbf{r}'(t)$$

by putting $K(t) = k \in \mathbb{R} \setminus \{0\}$.

7. (a) For any $t \in (0,1)$, since

$$\mathbf{r}'(t) = \left(1, -\frac{1}{t^2}\cos\left(\frac{1}{t}\right)\right) \neq (0,0)$$

Thus, this curve is regular.
Also, since

$$\|\mathbf{r}'(t)\| = \sqrt{1 + \left(-\frac{1}{t^2}\cos\left(\frac{1}{t}\right)\right)^2} \neq 1$$

as $\frac{1}{t^2}\cos^2\left(\frac{1}{t}\right) > 0$ for $t \in (0,1)$.
Thus, this curve is not parametrized by arc length.

(b) For positive integer k, we have

$$\begin{aligned}
\mathbf{r}\left(\left(2k\pi + \frac{\pi}{2}\right)^{-1}\right) &= \left(\left(2k\pi + \frac{\pi}{2}\right)^{-1}, \sin\left(2k\pi + \frac{\pi}{2}\right)\right) \\
&= \left(\frac{2}{(4k+1)\pi}, 1\right) \\
\mathbf{r}\left((2k\pi)^{-1}\right) &= \left((2k\pi)^{-1}, \sin(2k\pi)\right) \\
&= \left(\frac{1}{2k\pi}, 0\right)
\end{aligned}$$

(c) First, we define $I_k = \left[\frac{2}{(4k+1)\pi}, \frac{1}{2k\pi}\right]$ for any positive integers k.
Since

$$\begin{aligned}
\max_{k\in\mathbb{N}} \frac{1}{2k\pi} &= \frac{1}{2\pi} < 1 \\
\inf_{k\in\mathbb{N}} \frac{2}{(4k+1)\pi} &= 0 \geq 0
\end{aligned}$$

and each I_k is compact, we have

$$\bigcup_{k\in\mathbb{N}} I_k \subset (0,1).$$

Now, define $\mathbf{p}_k = \left(\frac{2}{(4k+1)\pi}, 1\right)$ and $\mathbf{q}_k = \left(\frac{1}{2k\pi}, 0\right)$ for each $k \in \mathbb{N}$.
Since the arc-length of $\mathbf{r}(t)$ over each I_k is

$$\begin{aligned}
\int_{|I_k|} \|\mathbf{r}'(t)\|\, dt &> \|\mathbf{p}_k - \mathbf{q}_k\| \\
&= \sqrt{1^2 + \left(\frac{1}{2k\pi} - \frac{2}{(4k+1)\pi}\right)^2} \\
&> 1
\end{aligned}$$

Thus, the arc-length of $\mathbf{r}(t)$ over $(0,1)$ is

$$\begin{aligned}
\int_{(0,1)} \|\mathbf{r}'(t)\| dt &> \int_{\bigcup_{k\in\mathbb{N}} I_k} \|\mathbf{r}'(t)\|\, dt \\
&= \sum_{k\in\mathbb{N}} \int_{|I_k|} \|\mathbf{r}'(t)\| dt \\
&> \sum_{k\in\mathbb{N}} (1) \to +\infty
\end{aligned}$$

Thus, $\mathbf{r}(t)$ has an infinite length over $(0,1)$.

8. (a) Since we want the cycloid lying on the xz-plane, so we may rotate it about the x-axis anti-clockwisely by an angle $\alpha = \frac{\pi}{2}$, so that we have

$$\begin{aligned} R_x\left(\frac{\pi}{2}\right)\begin{pmatrix}\theta - \sin\theta \\ 1-\cos\theta \\ 0\end{pmatrix} &= \begin{pmatrix}1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}\theta - \sin\theta \\ 1-\cos\theta \\ 0\end{pmatrix} \\ &= \begin{pmatrix}\theta - \sin\theta \\ 0 \\ 1-\cos\theta\end{pmatrix} \\ &= \mathbf{r}(\theta) \end{aligned}$$

(b) By direct computation, we have

$$\mathbf{r}'(\theta) = (1-\cos\theta, 0, \sin\theta)$$

and

$$\begin{aligned} \|\mathbf{r}'(\theta)\| &= \sqrt{(1-\cos\theta)^2 + \sin^2\theta} \\ &= \sqrt{1 - 2\cos\theta + (\cos^2\theta + \sin^2\theta)} \\ &= \sqrt{2(1-\cos\theta)} \\ &= \sqrt{2\cdot 2\sin^2\frac{\theta}{2}} \\ &= 2\sin\frac{\theta}{2} \quad (\because \text{ for } \theta \in (0, 2\pi), \sin\frac{\theta}{2} > 0) \end{aligned}$$

(c) Since

$$\langle \mathbf{r}'(\theta), \mathbf{e}_3\rangle = \sin\theta = \|\mathbf{r}'(\theta)\|\|\mathbf{e}_3\|\cos\varphi(\theta)$$

From part (a), we have $\|\mathbf{r}'(\theta)\| = 2\sin\frac{\theta}{2}$.
Therefore, we have

$$\begin{aligned} \cos\varphi(\theta) &= \frac{\sin\theta}{2\sin\frac{\theta}{2}} \\ &= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}} \\ &= \cos\frac{\theta}{2} \end{aligned}$$

As cos is bijective on $(0, \pi)$, so $\cos\varphi(\theta) = \cos\dfrac{\theta}{2}$ implies that $\varphi(\theta) = \dfrac{\theta}{2}$.

(d) (i) Note that

$$\begin{aligned} \mathbf{v}(\theta) &= R_y(\varphi(\theta))\mathbf{u}(\theta) \\ &= \begin{pmatrix}\cos\frac{\theta}{2} & 0 & \sin\frac{\theta}{2} \\ 0 & 1 & 0 \\ -\sin\frac{\theta}{2} & 0 & \cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}\cos\theta \\ \sin\theta \\ 0\end{pmatrix} \\ &= \begin{pmatrix}\cos\frac{\theta}{2}\cos\theta \\ \sin\theta \\ -\sin\frac{\theta}{2}\cos\theta\end{pmatrix} \end{aligned}$$

and thus

$$\begin{aligned}\langle \mathbf{v}(\theta), \mathbf{r}'(\theta)\rangle &= (1-\cos\theta)(\cos\frac{\theta}{2}\cos\theta) + 0 - \sin\theta(\sin\frac{\theta}{2}\cos\theta) \\ &= \cos\theta\cos\frac{\theta}{2} - \cos\theta\left(\cos\frac{\theta}{2}\cos\theta + \sin\frac{\theta}{2}\sin\theta\right) \\ &= \cos\theta\cos\frac{\theta}{2} - \cos\theta\cos\frac{\theta}{2} \\ &= 0\end{aligned}$$

for any $\theta \in (0, 2\pi)$.

(ii) In the question, it provides that

$$\gamma(\theta) = \underbrace{\mathbf{v}(\theta)}_{\text{Inclined helix with angle } \frac{\theta}{2}} + \underbrace{\mathbf{r}(\theta)}_{\text{rotated curve by the helix}}$$

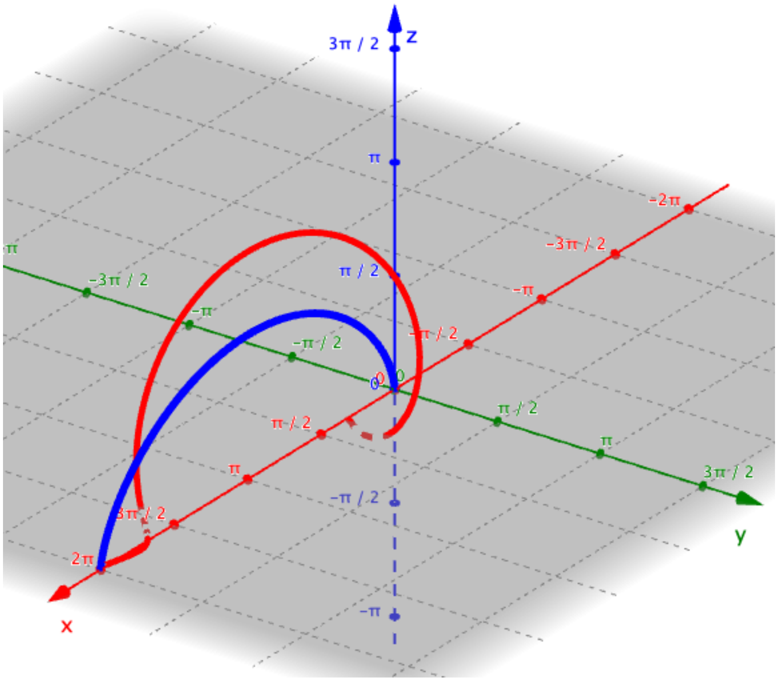

Figure 1: Rotated Helix around the cycloid

By part (d)(i), we have

$$\langle \gamma(\theta) - \mathbf{r}(\theta), \mathbf{r}'(\theta)\rangle = \langle \mathbf{v}(\theta), \mathbf{r}'(\theta)\rangle = 0$$

and this identity means the direction from $\mathbf{r}(\theta)$ to $\gamma(\theta)$ is always orthogonal to the tangential direction $\mathbf{r}'(\theta)$.

(iii) From part (d)(ii), we have

$$\begin{aligned}\gamma'(\theta) &= \mathbf{v}'(\theta) + \mathbf{r}'(\theta) \\ &= \frac{d}{d\theta}\begin{pmatrix} \frac{1}{2}\cos\frac{3\theta}{2} + \frac{1}{2}\cos\frac{\theta}{2} \\ \sin\theta \\ -\frac{1}{2}\sin\frac{3\theta}{2} + \frac{1}{2}\sin\frac{\theta}{2} \end{pmatrix} + \begin{pmatrix} 1-\cos\theta \\ 0 \\ \sin\theta \end{pmatrix} \\ &= \begin{pmatrix} -\frac{3}{4}\sin\frac{3\theta}{2} - \frac{1}{4}\sin\frac{\theta}{2} + 1 - \cos\theta \\ \cos\theta \\ -\frac{3}{4}\cos\frac{3\theta}{2} + \frac{1}{4}\cos\frac{\theta}{2} + \sin\theta \end{pmatrix}\end{aligned}$$

Therefore, we have

$$
\begin{aligned}
&\|\gamma'(\theta)\|^2 \\
&= \left(-\frac{3}{4}\sin\frac{3\theta}{2} - \frac{1}{4}\sin\frac{\theta}{2} + 1 - \cos\theta\right)^2 + \cos^2\theta + \left(-\frac{3}{4}\cos\frac{3\theta}{2} + \frac{1}{4}\cos\frac{\theta}{2} + \sin\theta\right)^2 \\
&= \underbrace{\left[\left(-\frac{3}{4}\sin\frac{3\theta}{2} - \frac{1}{4}\sin\frac{\theta}{2}\right)^2 + \left(-\frac{3}{4}\cos\frac{3\theta}{2} + \frac{1}{4}\cos\frac{\theta}{2}\right)^2\right]}_{(*)} + (1-\cos\theta)^2 \\
&- \left(\frac{3}{2}\sin\frac{3\theta}{2} + \frac{1}{2}\sin\frac{\theta}{2}\right)(1-\cos\theta) + \cos^2\theta + \left(-\frac{3}{2}\cos\frac{3\theta}{2} + \frac{1}{2}\cos\frac{\theta}{2}\right)\sin\theta + \sin^2\theta \\
&= \underbrace{\left[\frac{9}{16} - \frac{3}{8}\cos 2\theta + \frac{1}{16}\right]}_{(*)} + (1-\cos\theta)^2 - \left(\frac{3}{2}\sin\frac{3\theta}{2} + \frac{1}{2}\sin\frac{\theta}{2}\right) \\
&+ \frac{3}{2}\left(\sin\frac{3\theta}{2}\cos\theta - \cos\frac{3\theta}{2}\sin\theta\right) + \frac{1}{2}\left(\cos\frac{\theta}{2}\sin\theta + \sin\frac{\theta}{2}\cos\theta\right) + (\cos^2\theta + \sin^2\theta) \\
&= \frac{5}{8} - \frac{3}{8}\cos 2\theta + (1-\cos\theta)^2 - \left(\frac{3}{2}\sin\frac{3\theta}{2} + \frac{1}{2}\sin\frac{\theta}{2}\right) + \frac{3}{2}\sin\frac{\theta}{2} + \frac{1}{2}\sin\frac{3\theta}{2} + 1 \\
&= \frac{25}{8} + \frac{1}{8}\cos 2\theta - \sin\frac{3\theta}{2} - 2\cos\theta + \sin\frac{\theta}{2}
\end{aligned}
$$

and the last line is followed from $(1-\cos\theta)^2 = 1 - 2\cos\theta + \frac{1+\cos 2\theta}{2}$.
Thus, the arc length of $\gamma(\theta)$ is

$$
\begin{aligned}
\int_0^{2\pi} \|\gamma'(\theta)\|\, d\theta &= \int_0^{2\pi} \sqrt{\frac{25}{8} + \frac{1}{8}\cos 2\theta - \sin\frac{3\theta}{2} - 2\cos\theta + \sin\frac{\theta}{2}}\, d\theta \\
&= \frac{1}{2\sqrt{2}} \int_0^{2\pi} \sqrt{25 + \cos 2\theta - 8\sin\frac{3\theta}{2} - 16\cos\theta + 8\sin\frac{\theta}{2}}\, d\theta
\end{aligned}
$$